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DIRICHLET PROBLEM SOLUTION FOR SIMPLY AND DOUBLY CONNECTED DOMAINS WITH SMOOTH BOUNDARIES

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In this paper we propose a new method for solving the 2D Laplace equation with Dirichlet boundary conditions in simply and doubly connected domains. The method is based on reduction of the problem to the Fredholm integral equation of the second kind. The numerical algorithm connected with truncated Fourier series is applied to convert the Fredholm equation to a finite system of linear equations.

Keywords: Cauchy integral, Fourier polynomial, Dirichlet problem, Fredholm integral equation, simply connected domain, doubly connected domain.

Here we present Cauchy integral method of 2D Dirichlet problem solution for simply and doubly connected domain with smooth boundary. The method is based on reduction of the problem to the Fredholm integral equation of the second kind for the boundary values of the conjugate harmonic function and further conversion of the integral equation to the truncated linear system. The solution of the integral equation has the form of truncated Fourier series. Finally, the solution of the Dirichlet problem has the form of the real part of the Cauchy integral.

The case of simply connected domain

We denote the function to be found as $u(x,y) = \Re B(z)$, where $B(z)$ is analytic in a given simply connected domain Ω. So the problem follows: given the function $f_0(t) = u(x,y)|_{\{x(t),y(t)\}\in\partial\Omega}$, $t \in [0,2\pi]$, while the boundary smooth curve $\partial\Omega$ of the domain Ω passing in a counterclockwise direction, it is necessary to find the function $u(x,y)$, $(x,y)\in\Omega$.

By denoting $g_0(t) = \Im B(z(t))|_{z(t)=x(t)+iy(t)\in\partial\Omega}$, $t\in[0,2\pi]$, and separating the imaginary parts in both the sides of the criterion of the function $f_0(t)+ig_0(t)$ to be the boundary values of the function analytic in Ω, as in [2], the following Fredholm integral equation of the second kind is obtained:

$$g_0(t) = -\frac{1}{\pi}\int_0^{2\pi} f_0(\tau)\left(\log\left[z(\tau)-z(t)\right]\right)'_\tau d\tau + \frac{1}{\pi}\int_0^{2\pi} g_0(\tau)\left(\arg\left[z(\tau)-z(t)\right]\right)'_\tau d\tau. \qquad (1)$$

The integral equation (1) is similar to that appeared in [4] when the problem was solved applying the potential theory. The factor $(e^{i\tau}-e^{it})$ is extracted from the expression $(z(\tau)-z(t))$ in order to separate the improper VP integral in the Fredholm equation.

The solution is obtained in the form of truncated Fourier series $g_0(t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(nt) + \beta_n \sin(nt)$ by solving the following linear system:

$$\begin{pmatrix} AA & AB \\ BA & BB \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{2}$$

according to the following

Lemma. [1] *Let numbers $j, p > 1$ and a constant $U > 0$ exist such that $|\partial^{j+p} G(\tau, t)/\partial t^j \partial \tau^p| \leq U$ and the function $Y(t)$ possess the bounded second derivative: $Y''(t) < T$. Then, the approximate solution of the uniquely resolvable Fredholm integral equation of the second kind*

$$X(t) = \int_0^{2\pi} G(\tau, t) X(\tau) d\tau + Y(t),$$

where $Y(t)$ is 2π periodic and $G(\tau, t)$ is 2π periodic with respect to both variables, can be reduced to solution of finite linear system with error estimated by $\mathcal{O}\left(1/N^2\right)$ where N is the system rank.

After finding $g_0(t)$ we restore the Dirichlet problem solution as

$$u(x, y) = \Re \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_0(t) + i g_0(t)}{z(t) - x - iy} z'(t) dt.$$

The case of doubly connected domain

For the case of doubly connected domain Ω with boundary $\partial\Omega$, we assume that the boundary is composed of the outer smooth curve $L_0 : z_0(t), t \in [0, 2\pi]$, passed in counterclockwise direction and the inner smooth curve $L_1 : z_1(t), t \in [0, 2\pi]$, passed in clockwise direction. The Dirichlet problem follows: given the functions $f_j(t), j = 0, 1$, it is necessary to find $u(x, y), (x, y) \in \Omega$, such that $f_j(t) = u(x, y)|_{\{x(t), y(t)\} \in L_j}$ $, j = 0, 1,\ t \in [0, 2\pi]$. Assume that the solution of the Dirichlet problem, according to [2], is presented in the form $u(x, y) = \Re B(z) + A \log|z|, z = x + iy$, where A is a real-valued parameter which will be calculated later. So we have the boundary values of the analytic function $B(z)$ represented in the parametric form as follows:

$$B(z(t))|_{\partial\Omega} = \begin{cases} f_0(t) - A \log|z_0(t)| + i g_0(t), & z(t) \in L_0, \\ \\ f_1(t) - A \log|z_1(t)| + i g_1(t), & z(t) \in L_1. \end{cases} \tag{3}$$

Here $\{g_j(t)\}_{j=0}^{1} = \Im B(z(t))|_{z(t)=x(t)+iy(t) \in L_j}$, $t \in [0, 2\pi]$. By separating the imaginary parts of the corresponding integral equation as above, the following system of Fredholm integral equations is obtained:

$$
\begin{aligned}
g_0(t) = \frac{1}{\pi}\int_0^{2\pi} g_0(\tau)\left[\arg(z_0(\tau)-z_0(t))\right]'_\tau d\tau + \frac{1}{\pi}\int_0^{2\pi} g_1(\tau)\left[\arg(z_1(\tau)-z_0(t))\right]'_\tau d\tau \\
-\frac{1}{\pi}\int_0^{2\pi}\left(f_0(\tau)-A\log|z_0(\tau)|\right)\left[\log(z_0(\tau)-z_0(t))\right]'_\tau d\tau \\
-\frac{1}{\pi}\int_0^{2\pi}\left(f_1(\tau)-A\log|z_1(\tau)|\right)\left[\log(z_1(\tau)-z_0(t))\right]'_\tau d\tau
\end{aligned}
\quad (4)
$$

$$
\begin{aligned}
g_1(t) = \frac{1}{\pi}\int_0^{2\pi} g_0(\tau)\left[\arg(z_0(\tau)-z_1(t))\right]'_\tau d\tau + \frac{1}{\pi}\int_0^{2\pi} g_1(\tau)\left[\arg(z_1(\tau)-z_1(t))\right]'_\tau d\tau \\
-\frac{1}{\pi}\int_0^{2\pi}\left(f_0(\tau)-A\log|z_0(\tau)|\right)\left[\log(z_0(\tau)-z_1(t))\right]'_\tau d\tau \\
-\frac{1}{\pi}\int_0^{2\pi}\left(f_1(\tau)-A\log|z_1(\tau)|\right)\left[\log(z_1(\tau)-z_1(t))\right]'_\tau d\tau
\end{aligned}
\quad (5)
$$

The solution of the system with truncated Fourier series is obtained as a linear function of the constant A. Due to the properties of Cauchy integral, the parameter A can be easily calculated from the following formula:

$$
\sum_{s=0}^{1}\left(\int_0^{2\pi}\left[f_s(t)-A\log|z_s(t)|+ig_s(t)\right]\frac{\acute{z}_s(t)}{z_s^{k+1}(t)}dt\right)=0, \qquad k=0,1,\dots \quad (6)
$$

Finally, the Dirichlet problem solution $u(x,y)$ in the doubly connected domain Ω as a function of $z=(x+iy)$ can be expressed in the form

$$
u(z)=\Re\left(\sum_{j=0}^{1}\frac{1}{2\pi i}\int_0^{2\pi}\frac{f_j(t)-A\log|z_j(t)|+ig_j(t)}{z_j(t)-z}\acute{z}_j(t)dt\right)+\frac{A}{2}\log\left(x^2+y^2\right) \quad (7)
$$

The Cauchy integral method, applied to several examples, gave highly accurate results for irregular simply and doubly connected domains, it is also applicable for the domains with boundary components that are not star-like with respect to the origin [3], as presented in figure (1), for harmonic function with constant boundary values.

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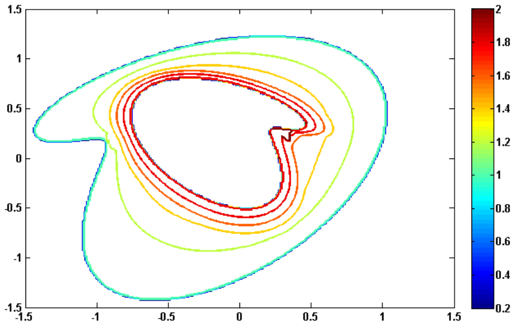

Fig. 1. The contour plot of the solution of the 2D Laplace equation in a domain with not star-like boundaries

РЕШЕНИЕ ЗАДАЧИ ДИРИХЛЕ ДЛЯ ОДНОСВЯЗНЫХ И ДВУСВЯЗНЫХ ОБЛАСТЕЙ С ГЛАДКИМИ ГРАНИЦАМИ

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В статье обсуждается новый метод решения двумерного уравнения Лапласа с граничными условиями Дирихле в односвязной и двусвязной области. Метод основан на сведении задачи к интегральному уравнению Фредгольма второго рода и применен численный алгоритм, связанный с усечением бесконечной системы линейных уравнений.

Ключевые слова: интеграл Коши, полином Фурье, задача Дирихле, интегральное уравнение Фредгольма, односвязная область, двусвязная область.

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CONTINUOUS ATOMIC SYSTEM FOR SUBSPACE

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In this talk we give a new contributions to atomic systems theory in Hilbert spaces. More precisely, we introduce and explain the concept of continuous atomic systems for subspaces, and give some examples to show differences between this and the discrete version.

Keywords: Hilbert space, continuous atomic systems, subspace.